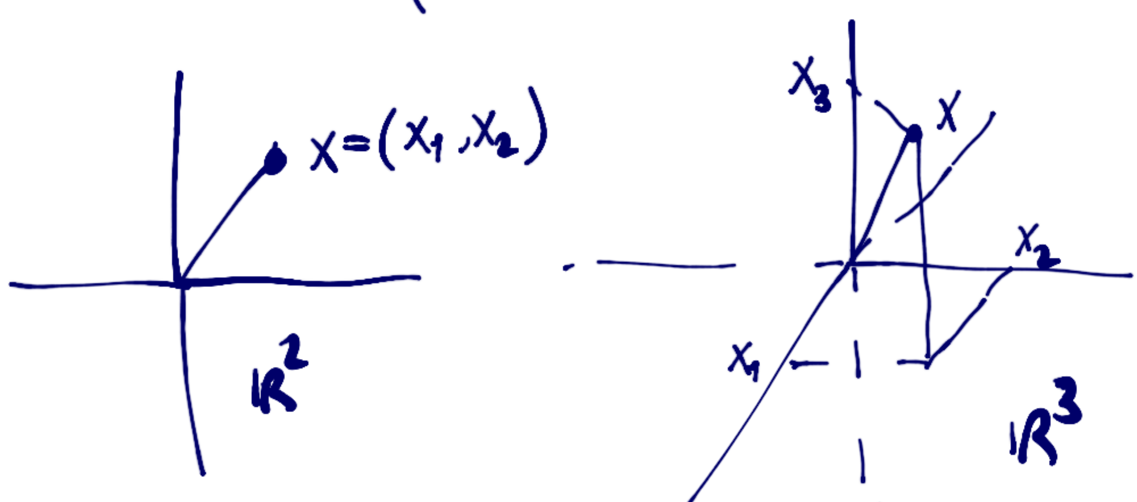
Euclidean space

Basic concepts

We define the Euclidean space $\mathbb{R}^N$, $N \geq 1$ using cartesian coordinates and geometrical concepts.

Each element $\boxed{(x_1, \dots, x_N)}$ of $\mathbb{R}^N$

$$\mathbf{x} = (x_1, \dots, x_N)$$



Given in the standard basis $\{e_1, \dots e_N\}$

$$x = \sum_{i=1}^{N} x_i e_i \quad , \quad x_i \in \mathbb{R}$$

In particular in $\mathbb{R}^3$, $B = \{ i, j, k \}$

<u>Properties</u> $\lambda, \mu \in \mathbb{R}$, $x, y \in \mathbb{R}^N$

Linear space.

a) Addition of vectors.

$$(x_1, \dots x_N) + (y_1, \dots y_N) = (x_1 + y_1, \dots x_N + y_N)$$

b) Multiplication by scalars, $\lambda \in \mathbb{R}$

$$\lambda (x_1, \dots x_N) = (\lambda x_1, \dots, \lambda x_N)$$

c) $(\mu \lambda)(x_1, \dots x_N) = \mu (\lambda x_1, \dots \lambda x_N))$

d) $(\mu + \lambda) x = \mu x + \lambda x$

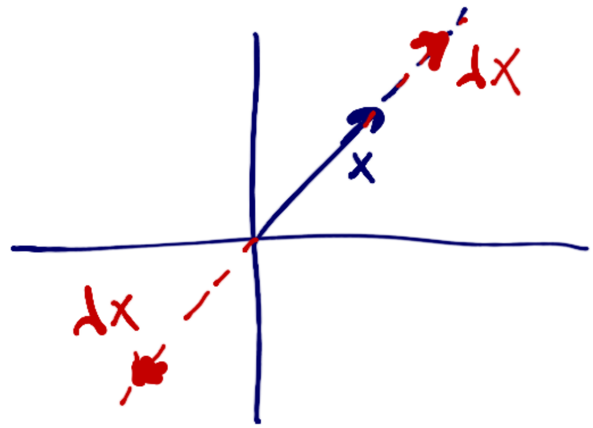
e) $\mu (x + y) = \mu x + \mu y$

f) $\underset{\in \mathbb{R}}{0} \cdot x = 0 \in \mathbb{R}^N$

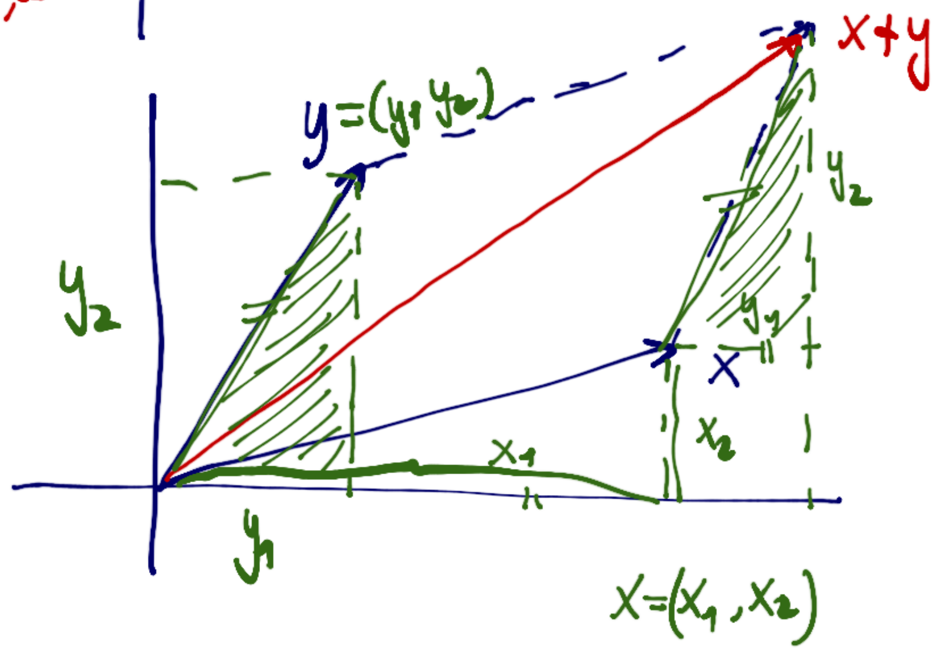
g) $1 \cdot x = x$

Geometrically

λx means a rescaling of the vector x



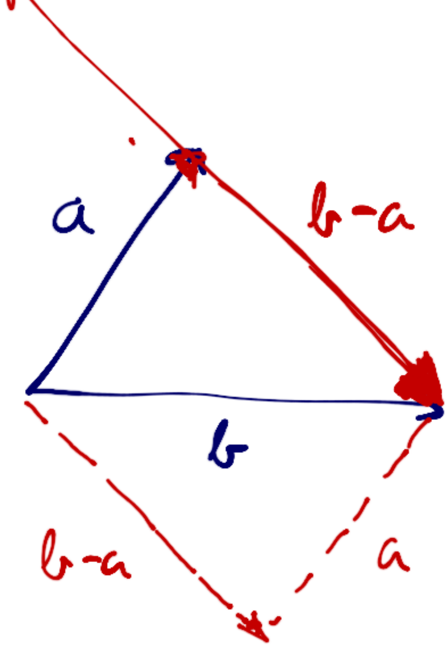
$x+y$



$$x+y=(x_1+x_2,\ y_1+y_2)$$

Remark

What is $b-a$? $a,b \in \mathbb{R}^N$



analytically

$a + (b-a) = b$

Property

The Euclidean space $\mathbb{R}^N$ is a normed space

It has an associated norm

$$\boxed{\|\cdot\| : \mathbb{R}^N \longrightarrow \mathbb{R}}$$

such that for any $x \in \mathbb{R}^N$

$$\boxed{\|x\| = \sqrt{x_1^2 + \dots + x_N^2} = \sqrt{\sum_{i=1}^{N} x_i^2}}$$

satisfying:

a) $\forall x \in \mathbb{R}^N \quad \|x\| > 0$ if $x \neq 0$

$\|x\| = 0$ if $x = 0$

b) $\|\lambda x\| = |\lambda| \, \|x\| \, , \ \forall \lambda \in \mathbb{R}, \ \forall x \in \mathbb{R}^N$

c) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^N$

Remark

Associated with the norm there exists the function "distance between two elements"

$$dist(x,y) = \|x-y\| \, , \quad dist : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}.$$

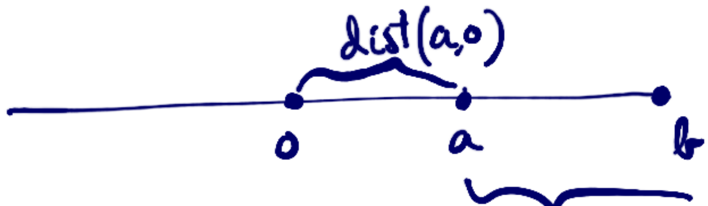
a) $dist(x,y) = \|x-y\| \geq 0$

b) $dist(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1| \, \|y-x\|$
$= dist(y,x)$

c) $dist(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|y-z\|$
$= dist(x,z) + dist(y,z)$

Remark.

In $\mathbb{R}$, dist ~ absolut value.

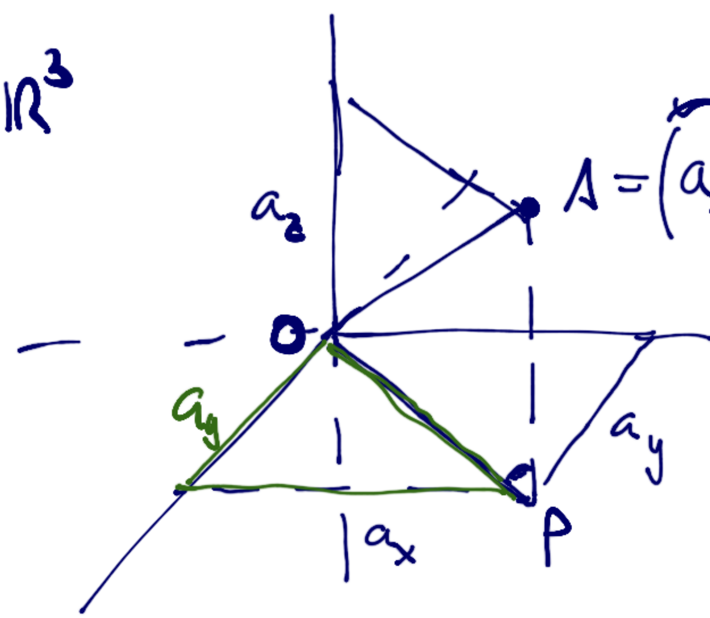


$$\operatorname{dist}(a,b) = |a-b|$$

In $\mathbb{R}^2$

$$\operatorname{dist}(x,0) = \|x\| = \sqrt{x_1^2 + x_2^2}$$

In $\mathbb{R}^3$



$$A = \overbrace{(a_x, a_y, a_z)}^{a}$$

$$OA = \operatorname{dist}(a,0) = \|a\|$$

$$\left.\begin{aligned} (OA)^2 &= \|a\|^2 = (OP)^2 + a_z^2 \\ (OP)^2 &= a_x^2 + a_y \end{aligned}\right\} \Rightarrow \|a\|^2 = a_x^2 + a_y^2 + a_z^2$$

Definition - Inner or scalar product

Let $a, b \in \mathbb{R}^n$ be two vectors in $\mathbb{R}^n$. Then the inner product is given by.

$$a \cdot b = a_1 b_1 + \cdots + a_n b_n$$
$$\|$$
$$\langle a, b \rangle = (a, b)$$

Remark $\quad \|x\| = \sqrt{\langle x, x \rangle} \equiv$ lenght of the vector x.

Properties

a) Positive definite $\quad \langle x, x \rangle > 0$ if $x \neq 0$

$\langle x, x \rangle = 0$ if $x = 0$

b) Symmetric $\quad \langle x, y \rangle = \langle y, x \rangle$

c) Bilineal $\quad \langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

Cauchy-Schwarz Inequality

$$|x \cdot y| \leq \|x\| \, \|y\|$$

Proof If $y = \lambda x$ (linearly dependent vectors)

$$|\langle x, \lambda x \rangle| = \left| \lambda \|x\|^2 \right| = |\lambda| \, \|x\|^2 = \|x\| \, \|y\|$$

$$\|y\| = |\lambda| \, \|x|$$

If $y \neq \lambda x$ (l.i vectors), we assume

$$z = \lambda x + y, \quad \lambda \in \mathbb{R}.$$

we know that

$$0 \leq \langle z, z \rangle = \langle \lambda x + y, \lambda x + y \rangle =$$

$$= \langle \lambda x, \lambda x \rangle + \langle \lambda x + y \rangle + \langle y, \lambda x \rangle + \langle y, y \rangle$$

$$= \lambda^2 \|x\|^2 + 2 \lambda \langle x, y \rangle + \|y\|^2$$

· Since y, x are l.i we cannot have real solutions for λ. and then,

$$4\left(\langle x,y\rangle\right)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\Downarrow$$

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|$$

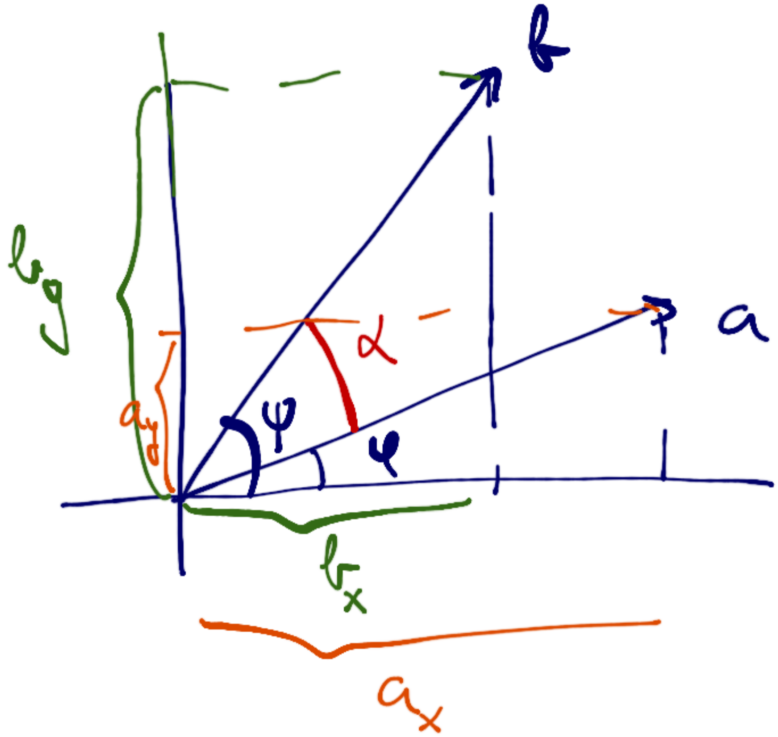
<u>Geometrical interpretation for the scalar product</u>

<u>Theorem</u>

$$\langle x,y\rangle = \|x\|\,\|y\| \cos\alpha$$

$\alpha \equiv$ angle between x and y.

Proof



$$a_x = \|a\| \cos\varphi \qquad a_y = \|a\| \sin\varphi$$

$$b_x = \|b\| \cos\psi \qquad b_y = \|b\| \sin\psi$$

$$\langle a, b\rangle = a_x b_x + a_y b_y = \|a\| \cos\varphi \, \|b\| \cos\psi + \|a\| \sin\varphi \, \|b\| \sin\psi$$

$$= \|a\| \, \|b\| \underbrace{\left(\cos\varphi \cos\psi + \sin\varphi \sin\psi\right)}_{\cos(\varphi - \psi) \,=\, \cos\alpha} .$$

Remark $\cos\alpha = \dfrac{\langle x, y\rangle}{\|x\| \, \|y\|}$ if $x \perp y \Rightarrow \langle x, y\rangle = 0$

Examples $C(a,b)$ = continuous functions in $[a,b]$

$f, g \in C(a,b)$

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt.$$

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_N y_N$$

We might have a weight

$$\langle f, g \rangle = \int_a^b w(t)\, f(t)\, g(t)\, dt.$$

such as

$$\langle f, g \rangle = \int_a^b e^{-t} f(t)\, g(t)\, dt.$$

Families of orthogonal functions

Solutions for Heat equ / Wave Eq. . . . $\{ \cos(nx), \sin(nx) \}$ in $[0, 2\pi]$

$$\int_0^{2\pi} \cos(nx)\sin(mx)\,dx = 0 = \int_0^{2\pi} \cos nx \cos mx \, dx$$

$n \neq m$

$$0 = \int_0^{2\pi} \sin nx \sin mx \, dx$$

$$\| \cos nx \|^2 = \int_0^{2\pi} \cos^2 nx \, dx = \pi = \| \sin nx \|^2$$

Vector product (only $\mathbb{R}^3$)

$\boxed{x \times y \in \mathbb{R}^3}$

$x, y \in \mathbb{R}^3$

$$x \times y = \overbrace{\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}^{\text{Notation}} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

Geometric interpretation

To do so, let's start with the triple product

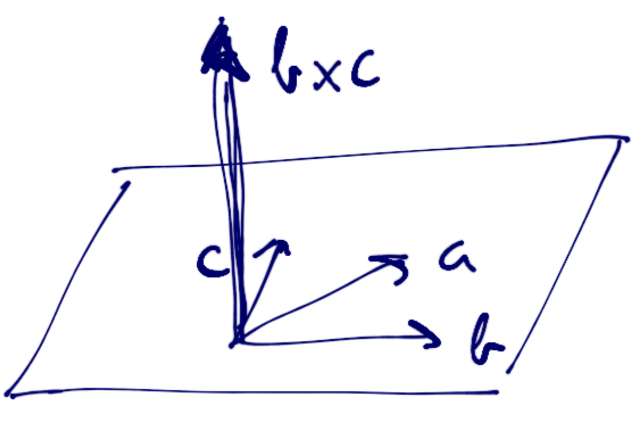
$$a\cdot(b\times c) = (a_1\, a_2\, a_3)\cdot\left(\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} i - \begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix} j + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix} k\right)$$

$$= a_1\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \Rightarrow \text{If } a \in \operatorname{span}\{b, c\}$$

$$\underbrace{a\cdot(b\times c) = 0}$$

$$\Downarrow$$



$$a \perp (b\times c)$$

Also

$$\boxed{\|b\times c\|^2} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}^2$$

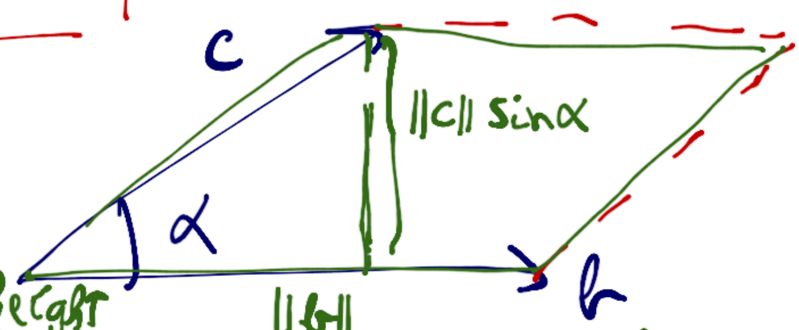
$$= (b_2c_3 - c_2b_3)^2 + (b_1c_3 - c_1b_3)^2$$

$$+ (b_1c_2 - c_1b_2)^2$$

$$= (b_1^2+b_2^2+b_3^2)(c_1^2+c_2^2+c_3^2) - (\underline{b_1c_1+b_2c_2+b_3c_3})^2$$

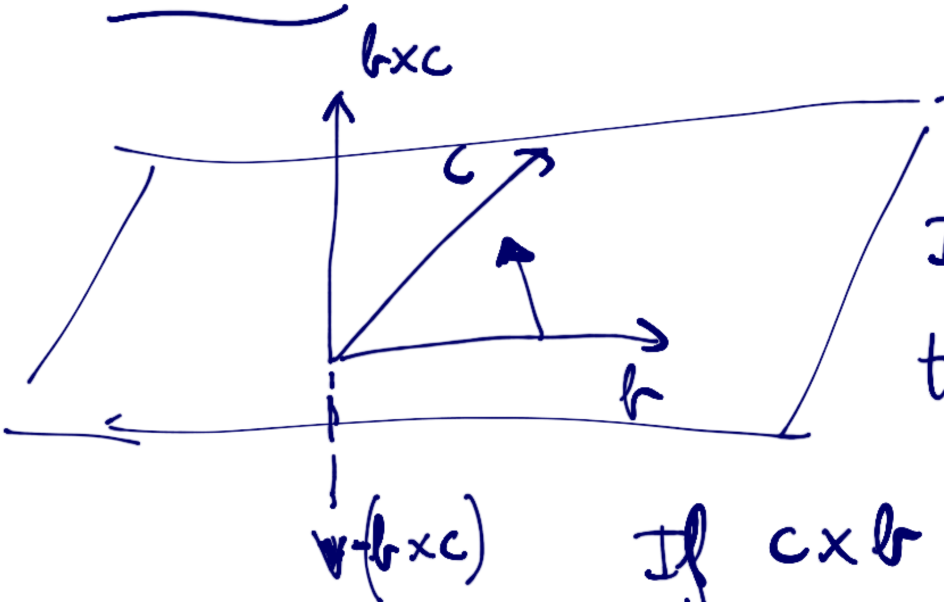
$$= \|b\|^2\|c\|^2 - (b\cdot c)^2 = 1\|b\|^2\|c\|^2 - \|b\|^2\|c\|^2\cos^2\alpha$$

$$= \boxed{\|b\|^2\|c\|^2\sin^2\alpha}$$



$$\|b\times c\| = \overbrace{\|b\|}^{\text{base.}}\ \overbrace{\|c\|\sin\alpha}^{\text{height}} \equiv \text{area of parallelogram.}$$

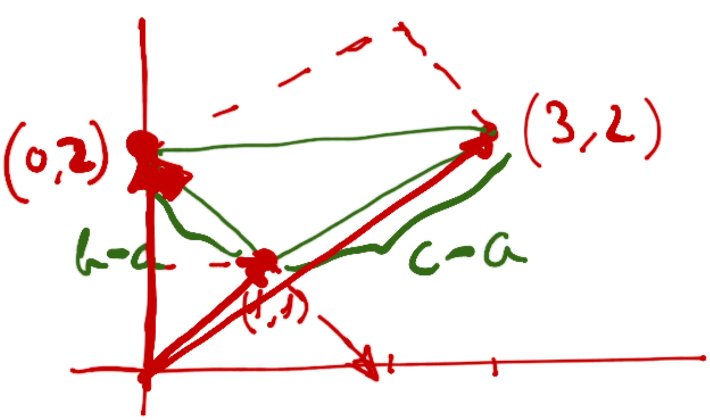
Remark



If $b \times c$ we choose the positive one.

If $c \times b = -(b \times c)$ we choose the negative orthog. vector.

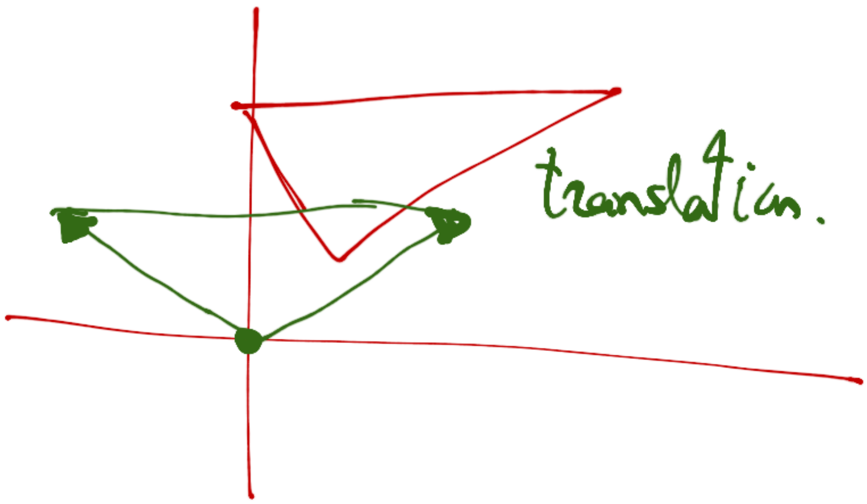
<u>Example</u>: Find the area of a triangle with vertexes on the points $(1,1)$, $(0,2)$ and $(3,2)$



Assume $\begin{cases} a = i + j \\ b = 2j \\ c = 3i + 2j \end{cases}$

Completing the parallelogram $A = \frac{1}{2}$ Area of parallelogram.

$$\Delta = \frac{1}{2} \| (b-a) \times (c-a) \| \equiv \text{Area of the triangle.}$$



Topology of $\mathbb{R}^N$

Structure of open sets.

Open set in $\mathbb{R}^N$: We define an open ball centered at the point $x_0 \in \mathbb{R}^N$ with radius $r > 0$ as

$$B(x_0, r) = B_r(x_0)$$

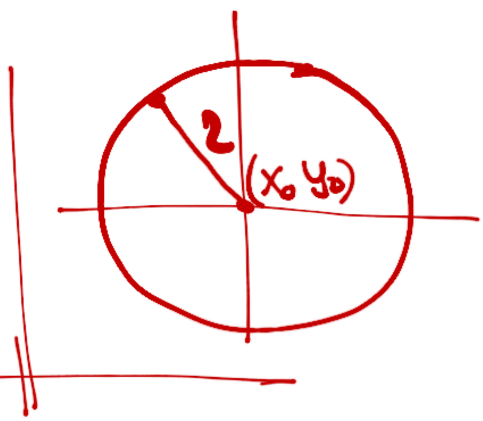
$$B(x_0, r) = \{ x \in \mathbb{R}^N : \text{dist}(x, x_0) = \|x - x_0\| < r \}$$

$$\|x-x_0\| = \sqrt{(x_1-x_{1_0})^2+\cdots+(x_N-x_{N_0})^2} < 2$$

$$(x_1-x_{1_0})^2+\cdots+(x_N-x_{N_0})^2 < 2^2$$

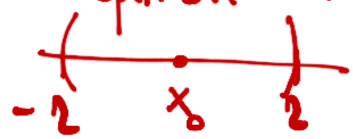
Ex: In $\mathbb{R}^2$ formula for a circumference.

$$(x-x_0)^2+(y-y_0)^2 = 2^2$$



In $\mathbb{R}$

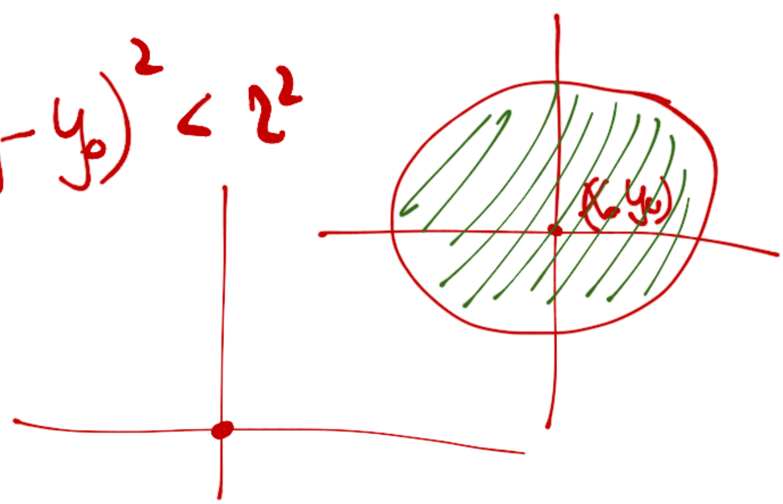
open interval.



$\boxed{|x|<2}$

Circle.

$$(x-x_0)^2+(y-y_0)^2 < 2^2$$



Closed ball

$$\overline{B(x_0, r)} = \{ x \in \mathbb{R}^N ; \ dist(x, x_0) \leq r \}$$

Boundary of the ball

$$\partial B(x_0, r) = \{ x \in \mathbb{R}^N ; \ dist(x, x_0) = r \}$$

Example : $\mathbb{R}^3$